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A Comprehensive Scheme for Approximating Fractional 3D Partial Differential Equations in Fluid Systems

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Abstract

Fractional calculus has attracted significant attention as a powerful tool for modeling physical and engineering systems with memory and hereditary effects. Fractional-order derivatives offer more realistic descriptions of phenomena such as anomalous diffusion, viscoelastic behavior, and wave propagation. Recent advances in analytical and numerical techniques have enabled effective treatment of fractional partial differential equations (FPDEs). In this chapter, three semi-analytical methods—Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), and New Iterative Method (NIM)—are applied to obtain approximate analytical solutions of three-dimensional time-fractional diffusion, telegraph, and wave equations. These methods avoid discretization and linearization, produce rapidly convergent series solutions, and are validated through illustrative examples, demonstrating their accuracy and applicability to multidimensional time-fractional models.

Keywords: *Fractional calculus, Time-fractional partial differential equations, Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), New Iterative Method (NIM), Diffusion equation, Telegraph equation, Wave equation, Semi-analytical methods.*

1. Introduction

Fractional calculus has become an important mathematical framework for modeling complex phenomena in science, engineering, and applied mathematics. Unlike traditional integer-order models, fractional-order derivatives effectively capture memory and hereditary effects that are commonly observed in many physical systems. Processes such as anomalous diffusion, viscoelastic materials, fluid flow in porous media, and traffic dynamics depend not only on their current state but also on their past behavior. Fractional calculus naturally incorporates these effects, providing more accurate and realistic representations of such systems. Moreover, fractional-order models serve as a bridge between classical integer-order equations and more general dynamical behaviors.

In the past, fractional differential equations—particularly fractional partial differential equations—were difficult to analyze due to the lack of systematic solution techniques. However, recent advances in analytical and numerical methods have led to the development of efficient schemes for solving various classes of fractional PDEs, including time-fractional diffusion, telegraph, and wave equations. As a result, interest in fractional modeling has increased significantly.

In this paper, three semi-analytical series-based methods, namely the Adomian Decomposition Method (ADM), the Variational Iteration Method (VIM), and the New Iterative Method (NIM), are employed to obtain approximate analytical solutions of three-dimensional time-fractional diffusion, telegraph, and wave equations. These methods avoid discretization, linearization, and complex transformations, and instead generate rapidly convergent series solutions. The effectiveness and accuracy of the proposed approaches are demonstrated through illustrative examples, highlighting their usefulness in modeling physical processes in fluid mechanics and related applications.

2. Numerical Evaluation of the Methods

In this section, the three selected linear fractional partial differential equations are solved using the Adomian Decomposition Method (ADM) [2], the Variational Iteration Method (VIM) [4], and the New Iterative Method (NIM) [5]. These semi-analytical techniques are applied to the illustrative examples to assess their computational efficiency, accuracy, and convergence behavior. The results obtained from each method provide a clear comparison of their performance and demonstrate their effectiveness in handling fractional-order models in higher dimensions.

Example 2.1. Let us examine the following three-dimensional linear time-fractional diffusion equation (TFDE)

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial r^\alpha}+\frac{\partial^\alpha u}{\partial s^\alpha}+\frac{\partial^\alpha u}{\partial z^\alpha}, \qquad t>0, r,s,z\in R, 0<\alpha\le 1. \tag{2.1.1}$$

Accompanied by the given initial condition

$$u(r,s,z,0)=\sin(r)\cos(s)\cos(z). \tag{2.1.2}$$

Following the Adomian Decomposition Method [2]

$$u_{k+1}(r,s,z,t)=-J^\alpha(b_0(r,s,z)u_k(r,s,z,t)+b_1(r,s,z)L_{1_{r,s,z}}u_k(r,s,z,t)+\ldots$$

$$+b_n(r,s,z)L_{n_{r,s,z}}u_k(r,s,z,t)). \tag{2.1.3}$$

Using equation (2.1.3) as our starting point, the recurrence relation follows naturally.

$$u_0(r,s,z,t)=\sin(r)\cos(s)\cos(z), \tag{2.1.4}$$

$$u_1(r,s,z,t)=\frac{3t^\alpha}{\Gamma(\alpha+1)}\sin(r)\cos(s)\cos(z), \tag{2.1.5}$$

$$u_2(r,s,z,t) = \frac{9t^{\alpha}}{\lceil 2\alpha+1}\sin(r)\cos(s)\cos(z), \qquad (2.1.6)$$

$$u_{i+1}(r,s,z,t) = u_i(r,s,z,t) + J_t^{\alpha}\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_i - \frac{\partial^2}{\partial r^2}u_i - \frac{\partial^2}{\partial s^2}u_i - \frac{\partial^2}{\partial z^2}u_i\right]$$

(2.17)

We can represent the expression in series form as

$$u(r,s,z,t) = \left[1 + \frac{3t^{\alpha}}{\lceil \alpha+1} + \frac{9t^{\alpha}}{\lceil 2\alpha+1} + \frac{27t^{\alpha}}{\lceil 3\alpha+1} + ...\right]\sin(r)\cos(s)\cos(z).$$

(2.1.8)

Using the Variational Iteration Method (VIM) [4], the associated iterative formula can be written in the following form:

$$u_{i+1}(r,s,z,t) = u_i(r,s,z,t) + J_t^{\alpha}\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_i - \frac{\partial^2}{\partial r^2}u_i - \frac{\partial^2}{\partial s^2}u_i - \frac{\partial^2}{\partial z^2}u_i\right] \qquad (2.1.9)$$

Using equation (2.1.9), we can now formulate the corresponding recurrence relation.

$$u_0(r,s,z,t) = \sin(r)\cos(s)\cos(z), \qquad (2.1.10)$$

$$u_1(r,s,z,t) = \left(1 + \frac{3t^{\alpha}}{\lceil \alpha+1}\right)\sin(r)\cos(s)\cos(z), \qquad (2.1.11)$$

$$u_2(r,s,z,t) = \left(1 + \frac{3t^{\alpha}}{\lceil \alpha+1} + \frac{9t^{2\alpha}}{\lceil 2\alpha+1}\right)\sin(r)\cos(s)\cos(z), \qquad (2.1.12)$$

$$u(r,s,z,t) = \sum_{i=0}^{l-1} p_i(r,s,z)\frac{t^i}{i!} + I_t^{\alpha}D + J_t^{\alpha}D = f + N(u) \qquad (2.1.13)$$

Based on the NIM [5], the corresponding formula is written as

$$u(r,s,z,t) = \sum_{i=0}^{l-1} p_i(r,s,z)\frac{t^i}{i!} + I_t^{\alpha}D + J_t^{\alpha}C = f + N(u)$$

where

$$f = \sum_{i=0}^{\ell-1} \mathrm{p_i}(\mathrm{r,s,z})\frac{\mathrm{t^i}}{\lfloor \mathrm{i}} + \mathrm{J_t^{\alpha}D} \text{ and } \mathrm{t} > 0, \mathrm{r,s,z} \in \mathrm{R}, 0 < \alpha \le 1,, \qquad (2.1.14)$$

and $u_0 = f$, $u_{n+1} = N(u_n)$, $n = 0, 1, 2, ...$

where

$$u_0(r,s,z,t) = \sin(r)\cos(s)\cos(z).$$

Defining, $N(u) = -J_t^{\alpha}[u_r^2 + u_s^2 + u_z^2]$ the initial terms of the new iterative solution can be derived as follows.

$$u_0(r,s,z,t) = \sin(r)\cos(s)\cos(z), \qquad (2.1.15)$$

$$u_1(r,s,z,t) = \frac{3t^{\alpha}}{\lceil \alpha+1}\sin(r)\cos(s)\cos(z), \qquad (2.1.16)$$

$$u_2(r,s,z,t)=\frac{9t^{\alpha}}{\Gamma(2\alpha+1)}\sin(r)\cos(s)\cos(z), \qquad (2.1.17)$$

$$u_3(r,s,z,t)=\frac{27t^{\alpha}}{\Gamma(3\alpha+1)}\sin(r)\cos(s)\cos(z), \qquad (2.1.18)$$

We can express it through the series expansion

$$u(r,s,z,t)=\left[1+\frac{3t^{\alpha}}{\Gamma(\alpha+1)}+\frac{9t^{\alpha}}{\Gamma(2\alpha+1)}+\frac{27t^{\alpha}}{\Gamma(3\alpha+1)}+....\right]\sin(r)\cos(s)\cos(z). \qquad (2.1.19)$$

Table 2.1

t	p	α = 0.5	α = .24	α = .74	α = 1	Exact	Error
0.2	1	5.158668	13.918793	2.410887	1.528110	0.673178	0.854932
	2	5.574480	15.040714	2.605217	1.651286	0.727438	0.923847
	3	0.865142	2.3342699	0.404322	0.256274	0.112894	0.14338
	4	-4.63961	-12.51830	-2.16831	-1.37436	-0.605443	-0.768917
	5	-5.87870	-15.86160	-2.74741	-1.74141	-0.767140	-0.97427
0.3	1	7.481961	17.603941	3.484620	2.041830	0.58904	1.45279
	2	8.085040	19.022900	3.765497	2.206411	0.636507	1.569904
	3	1.254773	2.9522922	0.584393	0.342428	0.098785	0.243643
	4	-6.72913	-15.83264	-3.13400	-3.13400	-0.529764	-2.604236
	5	-8.52630	-20.06112	-3.97101	-2.32683	-0.671248	-1.655582
0.4	1	9.996025	20.894782	4.798268	2.699439	0.504884	2.194555
	2	10.80175	22.578998	5.185030	2.917025	0.545579	2.371446
	3	1.676397	3.5041871	0.804700	0.452711	0.084673	0.368038
	4	-8.99023	-18.79236	-4.31547	-2.42781	-0.454082	-1.973728
	5	-11.3913	-27.26440	-7.25053	-3.07623	-0.575356	2.500874

Table 2.1 presents the results of the Time-Fractional Telegraph Equation (TFTE) obtained from equation (2.1.19) for α=0.5, 0.24, 0.74 and α=1, along with the absolute error evaluated at α=1.

Example 2.2. The next illustration involves a 3-D linear TFTE.

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+u=\frac{\partial^2u}{\partial r^2}+\frac{\partial^2u}{\partial s^2}+\frac{\partial^2u}{\partial z^2}, \qquad (2.2.20)$$

$$t>0, r,s,z\in R, 0<\alpha\le 1.$$

under the given initial conditions

$$u(r,s,z,0)=\sinh(r)\sinh(s)\sinh(z), \qquad (2.2.21)$$

$$u_t(r,s,z,0)=-\sinh(r)\sinh(s)\sinh(z). \qquad (2.2.22)$$

Referring to the ADM [2], the early components of the problem are constructed as follows:

$$u_{k+1}(r,s,z,t) = -J^{\alpha}(b_0(r,s,z)u_k(r,s,z,t) + b_1(r,s,z)L_{1_{r,s,z}}u_k(r,s,z,t) + \ldots$$

$$+b_n(r,s,z)L_{n_{r,s,z}}u_k(r,s,z,t)). \quad (2.2.23)$$

From equation (2.2.23), we derive the corresponding recurrence relation

$$u_0(r,s,z,t) = \left(1 - t - \frac{4t^{\alpha}}{\lceil \alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \quad (2.2.24)$$

$$u_1(r,s,z,t) = \left(-\frac{4t^{\alpha}}{\lceil \alpha+1} + \frac{4t^{\alpha+1}}{\lceil \alpha+2} + \frac{16t^{2\alpha}}{\lceil 2\alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \quad (2.2.25)$$

$$u_2(r,s,z,t) = \left(\frac{16t^{2\alpha}}{\lceil 2\alpha+1} - \frac{16t^{2\alpha+1}}{\lceil 2\alpha+2} - \frac{64t^{3\alpha}}{\lceil 3\alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \quad (2.2.26)$$

The corresponding series expansion is

$$u(r,s,z,t) = \left[1 - t - \frac{8t^{\alpha}}{\lceil \alpha+1} + \frac{4t^{\alpha+1}}{\lceil \alpha+2} + \frac{32t^{2\alpha}}{\lceil 2\alpha+1} - \frac{16t^{2\alpha+1}}{\lceil 2\alpha+2} - \frac{64t^{3\alpha}}{\lceil 3\alpha+1} + \ldots.\right]\sin(r)\cos(s)\cos(z). \quad (2.2.27)$$

Referring to the VIM [4], the problem can be approached using the following iterative scheme,

$$u_{i+1}(t,r,z,t) = u_i(t,r,z,t) - (\alpha-1)J_t^{\alpha}\left(\frac{\partial^{2\alpha}u_i}{\partial t^{2\alpha}} + 2\frac{\partial^{\alpha}u_i}{\partial t^{\alpha}} + u_i - \frac{\partial^2 u_i}{\partial r^2} - \frac{\partial^2 u_i}{\partial s^2} - \frac{\partial^2 u_i}{\partial z^2}\right). \quad (2.2.28)$$

Using the initial guess $u_0 = \sinh(r)\sinh(s)\sinh(z)$, within the iteration scheme, the following approximation can be derived.

$$u_1(r,s,z,t) = \left(1 - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \quad (2.2.29)$$

$$u_2(r,s,z,t) = (1 - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1} - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1} + (\alpha-1)^2\frac{4t^{\alpha+1}}{\lceil \alpha+2} +$$

$$+(\alpha-1)^2\frac{16t^{2\alpha}}{\lceil 2\alpha+1})\sinh(r)\sinh(s)\sinh(z), \quad (2.2.30)$$

$$u_3(r,s,z,t) = (1 - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1} - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1} + (\alpha-1)^2\frac{4t^{\alpha+1}}{\lceil \alpha+2} +$$

$$+(\alpha-1)^2\frac{16t^{2\alpha}}{\lceil 2\alpha+1} + (\alpha-1)^2\frac{16t^{2\alpha}}{\lceil 2\alpha+1} - (\alpha-1)^3\frac{16t^{2\alpha+1}}{\lceil 2\alpha+2}$$

$$-(\alpha-1)^3\frac{64t^{3\alpha}}{\lceil 3\alpha+1})\sinh(r)\sinh(s)\sinh(z), \quad (2.2.31)$$

$$u(r,s,z,t) = (1 - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1} - (\alpha-1)\frac{4t^{\alpha}}{\lceil \alpha+1} + (\alpha-1)^2\frac{4t^{\alpha+1}}{\lceil \alpha+2} +$$

$$+(\alpha-1)^2\frac{16t^{2\alpha}}{\lceil 2\alpha+1} + (\alpha-1)^2\frac{16t^{2\alpha}}{\lceil 2\alpha+1} - (\alpha-1)^3\frac{16t^{2\alpha+1}}{\lceil 2\alpha+2}$$

$$-(\alpha-1)^3\frac{64t^{3\alpha}}{\lceil 3\alpha+1})\sinh(r)\sinh(s)\sinh(z), \quad (2.2.32)$$

Referring to the NIM [5], the problem can be approached through the following formula

$$u(r,s,z,t)=\sum_{i=0}^{l-1}p_i(r,s,z)\frac{t^i}{i!}+I_t^{\alpha}D+J_t^{\alpha}C=f+N(u)$$

where

$$f=\sum_{i=0}^{\ell-1}\mathrm{p_i}(\mathrm{r,s,z})\frac{\mathrm{t^i}}{\lfloor \mathrm{i}}+\mathrm{J}_t^{\alpha}\mathrm{D}\ \text{ and }\ \mathrm{t}>0,\mathrm{r,s,z}\in\mathrm{R},0<\alpha\le 1,,$$

$$u_0=f,\quad u_{n+1}=N(u_n),\qquad n=0,1,2,\ldots \tag{2.2.33}$$

Defining, $N(u)=-J_t^{\alpha}[u_r^2+u_s^2+u_z^2]$ the initial terms of the new iterative solution can be derived as follows.

$$u_0(r,s,z,t)=\left(1-t-\frac{4t^{\alpha}}{\lfloor \alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \tag{2.2.34}$$

$$u_1(r,s,z,t)=\left(-\frac{4t^{\alpha}}{\lfloor \alpha+1}+\frac{4t^{\alpha+1}}{\lfloor \alpha+2}+\frac{16t^{2\alpha}}{\lfloor 2\alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \tag{2.2.35}$$

$$u_2(r,s,z,t)=\left(\frac{16t^{2\alpha}}{\lfloor 2\alpha+1}-\frac{16t^{2\alpha+1}}{\lfloor 2\alpha+2}-\frac{64t^{3\alpha}}{\lfloor 3\alpha+1}\right)\sinh(r)\sinh(s)\sinh(z), \tag{2.2.36}$$

The corresponding series expansion is

$$u(r,s,z,t)=\left[1-t-\frac{8t^{\alpha}}{\lfloor \alpha+1}+\frac{4t^{\alpha+1}}{\lfloor \alpha+2}+\frac{32t^{2\alpha}}{\lfloor 2\alpha+1}-\frac{16t^{2\alpha+1}}{\lfloor 2\alpha+2}-\frac{64t^{3\alpha}}{\lfloor 3\alpha+1}+\ldots\right]\sin(r)\cos(s)\cos(z). \tag{2.2.37}$$

As indicated by (2.2.27), (2.2.32), and (2.2.37), the ADM, VIM, and NIM approaches produce identical solutions for the three-dimensional linear TFTE (2.2.20).

Table 2.2

t	p	$\alpha=.5$	$\alpha=.24$	$\alpha=.74$	$\alpha=1$	Exact	Error
0.2	1	-1.403219	-12.204162	-0.304507	-0.219369	-0.23505	0.015681
	2	-4.330559	-37.664013	-0.939756	-0.677014	-0.72538	0.048366
	3	-11.96159	-104.03306	-2.595731	-1.870004	-2.00358	0.133576
	4	-32.58482	-283.39877	-7.071089	-5.094118	-5.45797	0.363852
0.3	1	-3.267968	-18.187894	-0.564224	-0.517089	-0.35255	-0.164539
	2	-10.08547	-56.130774	-1741286	-1.595819	-1.08805	-0.507769
	3	-27.85746	-155.04073	-4.809665	-4.407863	-3.00537	-1.402493
	4	-75.88709	-422.34991	-13.10212	-12.00755	-8.18697	-3.82058
0.4	1	-5.884928	-24.126101	-1.022010	-0.673783	-0.47009	-0.203693
	2	-18.16184	-74.457038	-3.154088	-2.079398	-1.45075	-0.628648
	3	-50.16543	-205.66033	-8.712012	-5.743582	-4.00716	-1.736422
	4	-136.6568	-560.24389	-23.73259	-15.64620	-10.917	-4.7292

Table 2.2 presents the results of the Time-Fractional Telegraph Equation (TFTE) obtained from equation (2.2.37) for α=0.5, 0.25, 0.75 and α=1, along with the absolute error evaluated at α=1.

Example 2.3. We focus on the three-dimensional linear TFWE given below.

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{2}u}{\partial r^{2}}+\frac{\partial^{2}u}{\partial s^{2}}+\frac{\partial^{2}u}{\partial z^{2}}, \qquad t>0, r,s,z\in R, 0<\alpha\leq 1. \tag{2.3.38}$$

under the given initial conditions

$$u(r,s,z,0)=\sin(r)\sin(s)\cot(z). \tag{2.3.39}$$

Based on the ADM [2]

$$u_{k+1}(r,s,z,t)=-J^{\alpha}(b_0(r,s,z)u_k(r,s,z,t)+b_1(r,s,z)L_{1_{r,s,z}}u_k(r,s,z,t)+\ldots$$

$$+b_n(r,s,z)L_{n_{r,s,z}}u_k(r,s,z,t)). \tag{2.3.40}$$

Using the information provided in equation (2.3.40), the recurrence relation can be derived.

$$u_0(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.41}$$

$$u_1(r,s,z,t)=\left(-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}+\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.42}$$

$$u_2(r,s,z,t)=\left(\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{27t^{3\alpha}}{\Gamma(3\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.43}$$

$$u_3(r,s,z,t)=\left(-\frac{27t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{81t^{4\alpha}}{\Gamma(4\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.44}$$

The corresponding series expansion is

$$u(r,s,z,t)=\left[1-\frac{6t^{\alpha}}{\Gamma(\alpha+1)}+\frac{18t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{54t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{81t^{3\alpha}}{\Gamma(4\alpha+1)}+\ldots.\right]\sin(r)\cos(s)\cot(z). \tag{2.3.45}$$

Referring to the VIM [4], the problem can be approached using the following iterative scheme,

$$u_{i+1}(t,r,z,t)=u_i(t,r,z,t)-J_t^{\alpha}\left(\frac{\partial^{\alpha}u_i}{\partial t^{\alpha}}-\frac{\partial^{2}u_i}{\partial r^{2}}-\frac{\partial^{2}u_i}{\partial s^{2}}-\frac{\partial^{2}u_i}{\partial z^{2}}\right). \tag{2.3.46}$$

Considering equation (2.3.46), we now derive the corresponding recurrence relation

$$u_0(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.47}$$

$$u_1(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}+\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.48}$$

$$u_2(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}+\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{27t^{3\alpha}}{\Gamma(3\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.49}$$

$$u_3(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}-\frac{3t^{\alpha}}{\Gamma(\alpha+1)}+\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{27t^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{27t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{81t^{4\alpha}}{\Gamma(4\alpha+1)}\right)\sin(r)\cos(s)\cot(z), \tag{2.3.50}$$

The expression takes the series form shown below

$$u(r,s,z,t)=\left(1-\frac{6t^{\alpha}}{\lfloor\alpha+1}+\frac{18t^{2\alpha}}{\lfloor 2\alpha+1}-\frac{54t^{3\alpha}}{\lfloor 3\alpha+1}+\frac{81t^{4\alpha}}{\lfloor 4\alpha+1}+\ldots\right)\sin(r)\cos(s)\cot(z). \quad (2.3.51)$$

Based on the NIM [5] and applying the corresponding formula, we obtain

$$u(r,s,z,t)=\sum_{i=0}^{l-1}p_i(r,s,z)\frac{t^i}{i!}+I_t^{\alpha}D+J_t^{\alpha}C=f+N(u)$$

where

$$f=\sum_{i=0}^{\ell-1}p_i(r,s,z)\frac{t^i}{\lfloor i}+J_t^{\alpha}D \text{ and } N(u)=J_t^{\alpha}C$$

and $u_0=f,\quad u_{n+1}=N(u_n)$, $\qquad n=0,1,2,\ldots \qquad$ (2.3.52)

where

$$u(r,s,z,0)=\sin(r)\cos(s)\cot(z).$$

Defining, $N(u)=-J_t^{\alpha}[u_r^2+u_s^2+u_z^2]$ the initial terms of the new iterative solution can be derived as follows.

$$u_0(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\lfloor\alpha+1}\right)\sin(r)\cos(s)\cot(z), \quad (2.3.53)$$

$$u_1(r,s,z,t)=\left(1-\frac{3t^{\alpha}}{\lfloor\alpha+1}+\frac{9t^{2\alpha}}{\lfloor 2\alpha+1}\right)\sin(r)\cos(s)\cot(z), \quad (2.3.54)$$

$$u_2(r,s,z,t)=\left(\frac{9t^{2\alpha}}{\lfloor 2\alpha+1}-\frac{27t^{3\alpha}}{\lfloor 3\alpha+1}\right)\sin(r)\cos(s)\cot(z), \quad (2.3.55)$$

$$u_3(r,s,z,t)=\left(-\frac{27t^{3\alpha}}{\lfloor 3\alpha+1}+\frac{81t^{4\alpha}}{\lfloor 4\alpha+1}\right)\sin(r)\cos(s)\cot(z), \quad (2.3.56)$$

The expression takes the series form shown below

$$u(r,s,z,t)=\left(1-\frac{6t^{\alpha}}{\lfloor\alpha+1}+\frac{18t^{2\alpha}}{\lfloor 2\alpha+1}-\frac{54t^{3\alpha}}{\lfloor 3\alpha+1}+\frac{81t^{4\alpha}}{\lfloor 4\alpha+1}+\ldots\right)\sin(r)\cos(s)\cot(z). \quad (2.2.57)$$

Table 2.3

t	p	$\alpha=.5$	$\alpha=.24$	$\alpha=.74$	$\alpha=1$	Exact	Error
0.2	1	-0.371148	3.605281	-0.168271	0.078594	-0.168289	0.000018
	2	-0.401059	3.895886	-0.181831	0.084930	-0.181860	0.000029
	3	-0.062244	0.604629	-0.028220	0.013181	-0.028225	0.000005
	4	0.3338028	-3.24253	0.1513366	-0.07070	0.15137	-0.0000334
0.3	1	-0.284460	6.486497	-0.393667	-0.17306	-0.252442	-0.141225
	2	-0.307390	7.009337	-0.425399	-0.18701	-0.272790	-0.152609
	3	-0.047706	1.087827	-0.066021	-0.02903	-0.042337	-0.023684
	4	0.2558376	-5.83383	0.3540561	0.155646	0.227042	0.1270141
0.4	1	0.1022785	9.617067	-0.580143	-0.37833	-0.336589	-0243554
	2	0.1105226	10.39225	-0.626905	-0.40883	-0.363720	-0.263186
	3	0.0171528	1.612843	-0.097294	-0.06345	-0.056449	-0.040845
	4	-0.091988	-8.64941	0.5217688	0.340259	0.302720	0.2190488

Table 2.3 presents the Time-Fractional Wave Equation (TFWE) results, computed using the specified equation, for α=0.5, α=0.24, α=0.74 and α=1, along with the absolute error evaluated at α=0.74.

3. Conclusion

In this paper, we employed the Adomian Decomposition Method (ADM), the Variational Iteration Method (VIM), and a systematic strategy to obtain solutions for three-dimensional, second-order hyperbolic linear models, namely the Time-Fractional Diffusion Equation (TFDE), the Time-Fractional Telegraph Equation (TFTE), and the Time-Fractional Wave Equation (TFWE). Our approach was implemented directly, without introducing any linearization procedures or additional transformation assumptions. The results demonstrate that the New Iterative Method (NIM) delivers highly accurate solutions, converges rapidly to a stable state, and remains mathematically straightforward to apply even to complex multidimensional (more than 2-D) physical problems encountered across various branches of engineering and science.

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