

The Menger g -Contraction Maps and Menger Asymptotically g -Contraction Maps in S - Menger Space with Applications

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ABSTRACT

We define Menger g -contraction and Menger asymptotically g -contraction maps with examples in this study. We have examined two coincidence point results in S -Menger space as an application of Menger g -contraction and Menger asymptotically g -contraction maps.

Key Words: S -Menger space, coincidence point, Menger g -contraction and Menger asymptotically g -contraction maps.

MSC: 54E40, 54H25.

1. Introduction

Karl Menger first proposed the idea of PM space in 1942. Although the notion of Theory of fixed points in metric spaces appears abstract, it has significant applications in several areas like the study of integral and differential equations [3,4,13].

Later, the study of generalized metric structures attracted considerable attention. Gahler [7,8] presented the idea of 2-metric spaces, while B.C. Dhage [1] proposed the notion of D-metric spaces. However, Z. Mustafa and B. Sims [17] later observed that several results related to Dhage's D-metric spaces were not valid. To overcome these difficulties, A novel generalized metric framework was presented by them known as the G-metric space.

In the direction of generalized metric spaces, more advancements were made. In [10] S. Sedghi and Shobe the notion of D^* -metric space was introduced as a generalization of G-metric space. Later, S-metric space was first proposed by Sedghi et al., which generalizes both D^* -metric and G-metric spaces. They also established several fundamental characteristics of S-metric spaces and proved some fixed-point theorems for self-mappings in this setting [9].

On the other hand, coincidence point theorems for contraction and asymptotic contraction mappings in fuzzy metric spaces were studied by B. S. Choudhury et al. [2], which stimulated further research on weaker and asymptotically contractive conditions. By combining probabilistic structures with generalized metric concepts, S -Menger spaces emerged as an important framework for research on fixed-point theory. In this direction, broadly Banach and Kannan type fixed point results in S -Menger spaces were obtained by Krishna Kanta Sarkar et al. [12], demonstrating the effectiveness of such hybrid structures. Numerous relevant findings have been found in the body of current literature in [14,15,16].

In this document, we offer the concepts of Menger g – contraction and Menger asymptotically g – contraction in S -Menger spaces. As an application of these concepts, two coincidence point theorems are established in S -Menger spaces with examples. The obtained results extend the work of B. S. Choudhury et al. [2] in S -Menger areas.

2. Preliminaries

Definition 2.1. [11] A map $T: [0,1]^3 \rightarrow [0,1]$ is a continuous t-norm if it meets the requirements listed below:

$$T_1: T(\xi, 1, 1) = \xi, T(0, 0, 0) = 0$$

$$T_2: T(\xi, \zeta, \eta) = T(\xi, \eta, \zeta) = T(\zeta, \eta, \xi)$$

$$T_3: T(\xi_1, \zeta_1, \eta_1) \geq T(\xi_2, \zeta_2, \eta_2) \text{ for } \xi_1 \geq \xi_2, \zeta_1 \geq \zeta_2, \eta_1 \geq \eta_2$$

examples of t-norm are

(1): $T_p(\xi, \zeta, \eta) = \xi \cdot \zeta \cdot \eta$ product t-norm;

(2): $T_m(\xi, \zeta, \eta) = \min\{\xi, \zeta, \eta\}$ minimum t-norm

Definition 2.2 [13] Let g and f be two self-mappings on a nonempty set X (i. e, $g, f: X \rightarrow X$). If $gz = fz$ for some $z \in X$. then z is called a coincidence point of g and f . The mapping g and f are said to be weakly – compatible if they commute at their coincident point, i.e., $(gf)z = (fg)z$.

Definition 2.3 S-Menger space [12]

S -Menger space is defined as the 3-tuple (X, S, T) . if X is a non-empty set, S is a function defined on X^3 to the set of distribution function and T is a continuous third order t-norm such that the following conditions are satisfied:

- (i) $S_{(\xi, \zeta, \eta)}(0) = 0$ for all $\xi, \zeta, \eta \in X$
- (ii) $S_{(\xi, \xi, \zeta)}(t) < 1$ for $t > 0$ with $\xi \neq \zeta$,
- (iii) $S_{(\xi, \zeta, \eta)}(t) = 1$ for all $t > 0$, if and only if $\xi = \eta = \zeta$.
- (iv) $S_{(\xi, \zeta, \eta)}(t) \geq T(S_{(\xi, \xi, p)}(t_1), S_{(\zeta, \zeta, p)}(t_2), S_{(\eta, \eta, p)}(t_3))$,

Where $t = t_1 + t_2 + t_3$ and $t, t_1, t_2, t_3 > 0$ for all $\xi, \zeta, \eta, p \in X$

Example 2.1. [12] Assume that $X = R$ is a real line. The S -metric on X is defined by

$$S_{(\xi, \zeta, \eta)}(t) = \begin{cases} |\xi - \eta| + |\zeta - \eta|, & t > 0. \\ 0, & t = 0 \end{cases}$$

Define $p, q, r = pqr$, $\forall p, q, r \in [0,1]$ and let S be the distribution function on $X^3 \times (0, \infty)$ define by

$$S_{(\xi, \zeta, \eta)}(t) = \frac{t}{t + S(\xi, \zeta, \eta)} \text{ for all } \xi, \zeta, \eta \in X \text{ and } t > 0.$$

Then (R, S, o) is a space on X that is S -metric.

Example 2.2. [12] Let S be an S -metric space as described in 2.1, and let $X = R$ be a real line.

$T(\xi, \zeta, \eta) = \xi \cdot \zeta \cdot \eta$ for every $\xi, \zeta, \eta \in [0, 1]$ and let S be the distribution function on X^3 defined by

$$S_{(\xi, \zeta, \eta)}(t) = \begin{cases} 0, & t = 0 \\ \exp\left[\frac{S((\xi, \zeta, \eta))}{t}\right]^{-1}, & t > 0 \end{cases} \text{ for all } \xi, \zeta, \eta \in X \text{ and } t > 0.$$

Then (R, S, T) is an S -Menger space.

Lemma 2.1. [12] $S_{(\xi, \xi, \zeta)}(\cdot)$ is non-decreasing for all ξ, ζ in X .

Definition 2.4. [12] A sequence $\{\xi_n\}$ in (X, S, T) is converges to $\xi \in X$ if $S_{(\xi_n, \xi_n, \xi)}(t) \rightarrow 1$ as $n \rightarrow \infty$ for each $t > 0$. i.e for each $\epsilon > 0$ and $t > 0 \exists n_0 \in N$ so that for everyone

$$n \geq n_0, S_{(\xi_n, \xi_n, \xi)}(t) > 1 - \epsilon \text{ or } S_{(\xi, \xi, \xi_n)}(t) > 1 - \epsilon.$$

Lemma 2.2. [12] Let (X, S, T) be an S -Menger space, where T is minimum (H -type) t -norm. Let $\{\xi_n\}$ be a series in X . If $\{\xi_n\}$ concludes at ξ and $\{\xi_n\}$ also converges to ζ then $\xi = \zeta$. If exists is unique, that is the limit of $\{\xi_n\}$.

Definition 2.5. [12] Let (X, S, T) be an S -Menger space and $\{\xi_n\}$ be a sequence in X is referred to as a Cauchy sequence, if for any $\epsilon > 0$ and $t > 0$ there exists $n_0 \in N$ such that

$$S_{(\xi_n, \xi_n, \xi_m)}(t) > 1 - \epsilon$$

or

$$S_{(\xi_m, \xi_m, \xi_n)}(t) > 1 - \epsilon \text{ for all } n, m \geq n_0.$$

Definition 2.6. [12] Let (X, S, T) be an S -Menger space. X is referred to as a complete S -Menger space if all of its Cauchy sequences are convergent.

Lemma 2.3. [12] Consider the S -Menger space (X, S, T) , where T is minimum (H -type) t -norm and $\{\xi_n\}$ be a sequence in X . If $\{\xi_n\}$ converges to ξ , then $\{\xi_n\}$ is a Cauchy sequence.

Definition 2.7. [5] Let f and g be self-mappings on S -Menger space (X, S, T) and $\{\xi_n\}$ be a sequence in X . f is said to be asymptotically regular at a point $\xi_0 \in X$ if

$$S_{(f^n(\xi_0), f^n(\xi_0), f^{n+1}(\xi_0))}(t) = 1, \quad \forall t > 0$$

Also, the sequence $\{\xi_n\}$ is said to be asymptotically regular with respect to the pair (f, g) if

$$\lim_{n \rightarrow \infty} \left(S_{(f^n(\xi_n), f^n(\xi_n), f^{n+1}(\xi_n))}(t) \right) = 1, \quad \forall t > 0$$

3. Main Result

In this section, we will give Menger g -contraction and Menger asymptotically g -contraction maps with examples and give applications in coincidence point theory.

Definition 3.1. Let f and g be two mappings that are specified on S -Menger space (X, S, T) with values into itself. Then, f is referred to be a Menger g -contraction if

$$t > 0 \text{ and } S_{(g\xi, g\xi, g\xi)}(t) > 1 - t$$

$$\Rightarrow S_{(f\xi, f\xi, f\xi)}(kt) > 1 - kt, \quad \text{where } k \in (0,1)$$

Example 3.1. Let the S -Menger space be (X, S, T) . which is defined as in example 2.1. Let f and g are self-maps on X defined by $f(\xi) = \frac{\xi}{2}, g(\xi) = \frac{\xi}{4}, \xi \in X$

Then f is Menger g -contraction maps with respect to g , (for $k = \frac{1}{2}$)

Definition 3.2. Let f, g and T be three self-maps defined on S -Menger space (X, S, T) . Assume that T is asymptotically regular at $\xi \in X$. If $t > 0, f$ is referred to as Menger asymptotically g -contraction with respect to T

$$S_{(gT^n\xi, gT^n\xi, gT^{n+1}\xi)}(t) > 1 - t$$

$$\Rightarrow S_{(fT^n\xi, fT^n\xi, fT^{n+1}\xi)}(kt) > 1 - kt \text{ where } t > 0 \text{ and } k \in (0,1)$$

Example 3.2. Let (X, S, T) be the S -Menger space which is defined in example 2.1. Let f, g and T are self-maps on X defined by $f(\xi) = \frac{\xi}{2}, g(\xi) = \frac{\xi}{4}$ and $T(\xi) = \frac{\xi}{8}, \xi \in X$.

In that case, f is Menger asymptotically g -contraction with regard to T and T is asymptotically regular at $\xi \in X$, (for $k = \frac{1}{2}$).

Proposition 3.1. Let g be a bijective mapping on (X, S, T) and f is g -contraction on the S -Menger space (X, S, T) then $g^{-1}f$ is a continuous mapping on (X, S_g, T) with value into itself, where $S_{g(\xi, \eta)}(t) = S_{(g\xi, g\xi, g\xi)}(t)$.

Proof. Let $\{\xi_n\}_{n \geq 1}$ be a series in X such that $\xi_n \rightarrow \xi \in X$ under S -Menger S_g .

$$\Rightarrow S_{(g\xi_n, g\xi_n, g\xi)}(t) \rightarrow 1 \text{ as } n \rightarrow \infty, \text{ for every } t > 0. \text{ By definition 3.1, it follows that,}$$

$$\Rightarrow S_{(f\xi_n, f\xi_n, f\xi)}(kt) \rightarrow 1 \text{ as } n \rightarrow \infty, \text{ for every } t > 0,$$

$$\Rightarrow S_{(gg^{-1}f\xi_n, gg^{-1}f\xi_n, gg^{-1}f\xi)}(kt) \rightarrow 1 \text{ as } n \rightarrow \infty, \text{ for every } t > 0,$$

$$\Rightarrow S_g(g^{-1}f\xi_n, g^{-1}f\xi_n, g^{-1}f\xi)(kt) \rightarrow 1 \text{ as } n \rightarrow \infty, \text{ for every } t > 0.$$

This demonstrates that $g^{-1}f$ is a continuous mapping with values into itself on (X, S_g, T) .

Application of Menger g -contraction maps

In this section we provide an application of Menger g -contraction map in S -Menger space.

Theorem 3.1 Let (X, S, T) be a complete S -Menger space. Let f, g, T be three self-mappings on X with g is bijective, f, T are g -contractive mappings and $fg^{-1}f = Tg^{-1}T$. Then f, g and T have a coincidence point. That is there exists $p \in X$ such that $fp = gp = Tp$.

Proof: For $t > 1$ and $\xi, \zeta \in X$ We've $S_{(g\xi, g\xi, g\zeta)}(t) > 1 - t$. Now, according to the definition of g -contraction

We can state, $S_{(f\xi, f\xi, f\zeta)}(kt) > 1 - kt$. Where $k \in (0, 1)$

Now, $S_{(f\xi, f\xi, f\zeta)}(kt) > 1 - kt$

$\Rightarrow S_{(gg^{-1}f\xi, gg^{-1}f\xi, gg^{-1}f\zeta)}(kt) > 1 - kt$

That is, $S_{g(g^{-1}f\xi, g^{-1}f\xi, g^{-1}f\zeta)}(kt) > 1 - kt$

$\Rightarrow S_{g(h\xi, h\xi, h\zeta)}(kt) > 1 - kt$ where $g^{-1}f = h$

Using the same technique, we obtain after the n^{th} iteration

$S_{g(h^n\xi, h^n\xi, h^n\zeta)}(k^nt) > 1 - k^nt$

For every $\varepsilon > 0, \delta \in (0, 1)$, there exist a positive $n(\varepsilon, \delta)$ such that $k^nt \leq \min(\varepsilon, \delta)$ for every $n \geq n(\varepsilon, \delta)$.

Now $S_{g(h^n\xi, h^n\xi, h^n\zeta)}(\varepsilon) \geq S_{g(h^n\xi, h^n\xi, h^n\zeta)}(k^nt) > 1 - k^nt > 1 - \delta$

Therefore $S_{g(h^n\xi, h^n\xi, h^n\zeta)}(\varepsilon) > 1 - \delta$

Let $\xi_0 \in X$ and $\{\xi_n\}$ be the sequence of consecutive approximations described as $\xi_{n+1} = h\xi_n$.

Taking $\xi = \xi_{m-n}$ and $\zeta = \xi_0$ from last inequality we get

$S_{g(\xi_m, \xi_m, \xi_n)}(\varepsilon) > 1 - \delta$ for every $n \geq n(\varepsilon, \delta)$ and $m \geq n$.

Therefore $\{\xi_n\}$ is Cauchy seq'n in X .

Since (X, S, T) is complete, then (X, S_g, T) is likewise finished and has a point $p \in X$ such that the sequence $\{\xi_n\}$ converges to p . Since, h is continuous on (X, S_g, T) we have, $hp = p$ this implies that $g^{-1}fp = p$ that is $fp = gp$.

p is a point of coincidence.

Again $t > 0$ and $S_{(g\xi, g\xi, g\zeta)}(t) > 1 - t$

$\Rightarrow S_{(T\xi, T\xi, T\zeta)}(kt) > 1 - kt, \quad k \in (0, 1)$

Likewise, we may demonstrate that there $q \in X$ st, $Tq = gq$.

We can prove $p = q$ by using $fg^{-1}f = Tg^{-1}T$,

Therefore, $fp = Tp = gp$,

That is, f, g and T have a coincidence point.

Application of Menger asymptotically g -contraction maps

In this section we prove following theorem to provide application of Menger asymptotically g -contraction map.

Theorem 3.2 Let f, g, T are three mappings that are defined on the entire S -Menger space (X, S, T) . with values into itself where g is bijective T is asymptotically regular at $\xi \in X$ and f is Menger asymptotically g -contraction with respect to T with $fT = Tf$, $gT = Tg$, $\xi \in X$ and $g^{-1}fT$ is continuous in X . Then $fT\xi = g\xi$.

Proof: Given that f is Menger's asymptotic g -contraction with respect to T , we obtain for $t > 0$ using definition 3.2.

$$\begin{aligned}
 & S_{(gT^n\xi, gT^n\xi, gT^{n+1}\xi)}(t) > 1 - t \\
 \Rightarrow & S_{(fT^n\xi, fT^n\xi, fT^{n+1}\xi)}(kt) > 1 - kt \text{ where } t > 0 \text{ and } k \in (0,1) \\
 \Rightarrow & S_{(gg^{-1}fT^n\xi, gg^{-1}f\xi, gg^{-1}fT^{n+1}\xi)}(kt) > 1 - kt \\
 \Rightarrow & S_{g(g^{-1}fT^n\xi, g^{-1}fT^n\xi, g^{-1}fT^{n+1}\xi)}(kt) > 1 - kt \\
 \Rightarrow & S_{g(g^{-1}fTT^{n-1}\xi, g^{-1}fTT^{n-1}\xi, g^{-1}fTT^n\xi)}(kt) > 1 - kt \\
 \Rightarrow & S_{g(AT^{n-1}\xi, AT^{n-1}\xi, AT^n\xi)}(kt) > 1 - kt, \text{ putting } g^{-1}fT = A \\
 \Rightarrow & S_{(gAT^{n-1}\xi, gAT^{n-1}\xi, gAT^n\xi)}(kt) > 1 - kt \\
 \Rightarrow & S_{(fAT^{n-1}\xi, fAT^{n-1}\xi, fAT^n\xi)}(k^2t) > 1 - k^2t \\
 \Rightarrow & S_{(fg^{-1}fTT^{n-1}\xi, fg^{-1}fTT^{n-1}\xi, fg^{-1}fTT^n\xi)}(k^2t) > 1 - k^2t \\
 \Rightarrow & S_{(fg^{-1}fTTT^{n-2}\xi, fg^{-1}fTTT^{n-2}\xi, fg^{-1}fTTT^{n-1}\xi)}(k^2t) > 1 - k^2t \\
 \Rightarrow & S_{(fTg^{-1}fTT^{n-2}\xi, fTg^{-1}fTT^{n-2}\xi, fTg^{-1}fTT^{n-1}\xi)}(k^2t) > 1 - k^2t \\
 \Rightarrow & S_{(gg^{-1}fTg^{-1}fTT^{n-2}\xi, gg^{-1}fTg^{-1}fTT^{n-2}\xi, gg^{-1}fTg^{-1}fTT^{n-1}\xi)}(k^2t) > 1 - k^2t \\
 \Rightarrow & S_{g(A^2T^{n-2}\xi, A^2T^{n-2}\xi, A^2T^{n-1}\xi)}(k^2t) > 1 - k^2t \text{ putting } g^{-1}fT = A
 \end{aligned}$$

Continuing this process, we get

$$S_{g(A^n\xi, A^n\xi, A^nT\xi)}(k^nt) > 1 - k^nt$$

For every $\varepsilon > 0, \delta \in (0,1)$, there exist a positive $n(\varepsilon, \delta)$ such that $k^nt \leq \min(\varepsilon, \delta)$ for every $n \geq n(\varepsilon, \delta)$.

$$\begin{aligned}
 & S_{g(A^n\xi, A^n\xi, A^nT\xi)}(\varepsilon) > S_{g(A^n\xi, A^n\xi, A^nT\xi)}(k^nt) > 1 - k^nt > 1 - \delta \\
 \Rightarrow & S_{g(A^n\xi, A^n\xi, A^nT\xi)}(\varepsilon) > 1 - \delta
 \end{aligned}$$

Let $\{\xi_n\}$ be a sequence defined by $\xi_{n+1} = A\xi_n$. If we take $T\xi = \xi_{m-n}$ and $\xi = \xi_0$ we get from the above inequality,

$$S_{g(\xi_m, \xi_m, \xi_n)}(\varepsilon) > 1 - \delta \text{ for every } n \geq n(\varepsilon, \delta) \text{ and } m \geq n.$$

Thus $\{\xi_n\}$ is a Cauchy sequence. (X, S_g, T) is complete since (X, S, T) is complete. Consequently, the sequence $\{\xi_n\}$ converges to $\xi \in X$.

Once more since A is constant on (X, S_g, T) , $A\xi = \xi$,

$$\Rightarrow g^{-1}fA\xi = \xi$$

That is, $fA\xi = g\xi$

Example 3.3. Let $X = [0, \infty)$ with $T(\xi, \varsigma, \eta) = \xi \cdot \varsigma \cdot \eta$ and let

$$S_{(\xi, \varsigma, \eta)}(t) = \begin{cases} \left[\exp\left(\frac{|\xi - \varsigma| + |\varsigma - \eta|}{t}\right) \right]^{-1} & , t > 0 \\ 0 & , t = 0 \end{cases} \text{ for all } \xi, \varsigma, \eta \in X \text{ and } t \in (0, \infty).$$

Consequently, (X, S, T) is a complete S -Menger space. Describe $f, g, T: X \rightarrow X$ by $f\xi = g\xi = \frac{\xi^3}{4}$ and $T\xi = \frac{\xi^4}{8}$, for all $\xi \in X$.

Then this example supports the Theorem 3.2.

Conclusion

In the current investigation, we defined Menger g -contraction maps, Menger asymptotically g -contraction maps in newly defined S -Menger space with examples and applications in coincidence point theory.

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