Pi International Journal of Mathematical Sciences
Volume 2 | Issue 2 | April 2026 | ISSN: 3107–9830 (Online)
Available Online at : https://pijms.com
DOI: https://doi.org/10.59828/pijms.v2i2.29

On the Accuracy and Efficiency of Galerkin and Direct Methods for Second-Order Boundary Value Problems with Neumann Boundary Conditions

Pavitra [1], **P. Pandey** [2]

[1]Research Scholar, Department of Mathematics, K.S. Saket P.G. College, Ayodhya, Uttar Pradesh, India
[2]Assistant Professor, Department of Mathematics, K.S. Saket P.G. College, Ayodhya, Uttar Pradesh, India

[1]**Corresponding Author Email:** pavitra@kssaketpgcollege.ac.in

Received: 03 March 2026 | **Accepted:** 05 April 2026 | **Published:** 29 April 2026

Abstract

This study presents a numerical analysis of a second-order mixed boundary value problem with Neumann boundary condition. The problem is first formulated and solved using the Galerkin method, where the Neumann condition is naturally incorporated through weak formu-lation. To independently verify the Galerkin solution, a direct numerical approach based on a shooting technique combined with a DMVS adaptive integrator is employed, and the unknown initial derivative is determined using a secant iteration. A detailed comparison between the analytical solution, the Galerkin approximation, and the direct numerical results is carried out in terms of solution profiles, initial slope, and error norms. The numerical results demonstrate excellent agreement between both approaches, confirming the accuracy, stability, and reliability of the Galerkin formulation. The study highlights that while the direct method offers computa-tional efficiency, the Galerkin method provides a systematic and robust framework suitable for mixed boundary value problems.

keywords: *Galerkin Method; Direct Numerical Method; Neumann Boundary Condition; Mixed Boundary Value Problem; Shooting Technique*

Mathematics Subject Classification (2020): *65L60, 65L10, 35J25.*

1 Introduction

Neumann boundary conditions(BCs) occur frequently in many physical and engineering models where the flux or derivative of the unknown function is prescribed at the boundary, such as in heat conduction, diffusion, and fluid flow problems [8]. These conditions naturally reflects conservation laws and energy balance in many real-world systems. Accurate numerical treatment of Neumann boundaries is therefore essential for obtaining reliable and physically meaningful solutions.

This work is motivated by the direct second-order method proposed by Phang et al. [5], which combines the efficiency of direct numerical integration with the robustness of the shooting technique and secant type updates. The unknown initial slope is ieratively adjusted using root-finding strategies such as the Newton or secant method until the boundary conditions are satisfied. A variable step-size strategy is employed to control the local truncation error and enhance numerical accuracy and stability.

The approach avoids reformulating the problem into a Variational form or employing basis expansions, and instead treats the second-order differential equation in its original form. By coupling the direct method with shooting and secant iteration, mixed boundary value problems are transformed into initial value problems, ensuring fast convergence and high accuracy. This hybrid framework provides a complementary and computationally efficient alternative to classical weighted residual and Galerkin-based approaches [6]. The numerical solution of second-order boundary value problems with Neumann boundary conditions plays an important role in several physical and engineering applications. Although such problems can be solved using different numerical approaches, their performance may vary significantly in terms of accuracy, convergence behavior, and computational efficiency, depending on the nature of the problem and discretization strategy. The Galerkin method, being a classical varitional approach with global basis representations, is well known for its stability and accuracy in boundary value problems.

In this paper, a comparative numerical study of the Galerkin method and a direct integration-based approach is presented. The notations were discussed in section 2 of this paper. The mathematical formulation was presented in section 3. Numerical examples were provided in section 4. A detailed comparison of both methods is presented in section 5. Finally, the conclusion in this paper is provided in section 6.

2 Notations

Symbol	Description
$u_G(x)$	Numerical solution obtained using the Galerkin method
$u_D(x)$	Numerical solution obtained using the direct method
h	Step size
N_G	Number of basis functions in the Galerkin method
N_D	Number of grid points in the direct method
$\|E_G\|_\infty$	Maximum error norm for Galerkin method
$\|E_D\|_\infty$	Maximum error norm for direct method
$\|E_G\|_2$	Discrete L_2–error norm for Galerkin method
$\|E_D\|_2$	Discrete L_2–error norm for direct method
TOL	Prescribed error tolerance
DMVS	Direct method variable step size adapted with shooting technique via secant method

3 Mathematical Formulation

In general, Consider a second-order linear differential equation of the form

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u-r(x)=0,\ a\leq x\leq b \tag{1}$$

with the boundary conditions,

$$\alpha_0 u(a)+\alpha_1 u'(a)=c_1$$

$$\beta_0 u(a)+\beta_1 u'(a)=c_2 \tag{2}$$

Where $\alpha_0,\beta_0,\alpha_1,\beta_1,c_1,c_2$ are constant and $p(x),q(x),r(x)$ are continuous functions. Such problems arise frequently in mathematical physics and engineering applications.

The given second-order boundary value problem is first transformed into an equivalent system of first-order ordinary differential equations by introducing suitable auxiliary variables. The resulting system is then solved numerically using a predictor-corrector integration scheme with adaptive step size control. The local truncation error is monitored to ensure stability and accuracy of the computed solution.

4 Numerical Examples

This section presents representative numerical examples to asses the accuracy, convergence behavior, and computational efficiency of the proposed direct method. The test problems are taken from our earlier work [4], allowing a direct and fair comparison with the Galerkin-based approach. All numerical computations and graphical plots are generated using MATLAB R2021a.

4.1 Example 1

4.1.1 Problem Definition

We consider the mixed boundary value problem

$$-u''(x)+u(x)=x^2,\ 0<x<1, \tag{3}$$

with boundary condition

$$u(0)=1,\ u'(1)=0. \tag{4}$$

The exact solution is given by

$$u(x)=x^2-\frac{e^x(2e+1)}{(e^2+1)}+\frac{e^{-x}(2e-e^2)}{(e^2+1)}+2.$$

In our earlier work [4], the same boundary value problem was solved using a Galerkin-based shooting approach. During that analysis, the unknown initial derivative was computed numerically. This value is reused here solely for comparison purposes, ensuring that both methods start from identical initial information and is given by

$$u'(0) \approx -0.534518.$$

In the present study, this value is directly employed as the initial condition for the direct method with variable step size, thereby converting the boundary value problem, which is then numerically integrated over the domain.

4.1.2 Conversion to First-Order System

Let

$$y_1(x) = u(x),\ y_2(x) = u'(x)$$

$$y_1'(x) = y_2(x)$$

So,

$$y'(x) = y_1(x) - x^2, 0 < x < 1$$

$$y_1(0) = 1,\ y_2(0) \approx -0.534518.$$

4.1.3 Numerical Integration

The first-order system is solved using a predictor-corrector scheme. The predictor provides an explicit estimate of the solution, which is refined using an implicit corrector to improve accuracy. An adaptive step size strategy based on the local truncation error is employed.

4.1.4 Results and Discussion

The numerical results demonstrate an excellent agreement between the exact and numerical solutions over the entire interval $[0, 1]$. The two curves are nearly indistinguishable, confirming the high accuracy of the proposed direct method. The small magnitude of the absolute error further validates the reliability of the numerical integration strategy.

Figure 1 illustrates the comparison between the exact solution and the numerical solution obtained using the direct method(DMVS) over the interval $[0, 1]$. The two curves are almost indistinguishable throughout the domain, demonstrating the high accuracy of the proposed scheme.

Figure 2 presents the pointwise absolute error defined as

$$e(x) = |u_{exact}(x) - u_{DMVS}(x)|.$$

The error remains very small across the entire domain, with the maximum error corresponding to the L_∞ norm.

Figure 3 shows the variation of the numerical error with respect to the step size. The bounded and smooth nature of the curve indicates numerical stability of the method. The decrease in error with finer step sizes confirms the convergence behavior of the direct scheme.

4.2 Example 2

Consider the second-order linear ODE

$$\frac{d^2u}{dx^2} = u + x \tag{5}$$

with the prescribed boundary conditions

$$u(0) = 1,\ u(1) = 2. \tag{6}$$

The exact solution is given by

$$u(x) = \frac{3e-1}{e^2-1}e^x + \frac{e(e-3)}{e^2-1}e^{-x} - x.$$

In this example, the required initial derivative $u'(0) = 0.23972$ has already been determined using the shooting method in the previous analysis. Therefore, the boundary value problem is directly converted into an equivalent initial value problem by using this computed initial slope.

Once the initial conditions are known, the problem is solved following the same procedure as in Example1.

Figure 4-6 present the numerical results for Example 2 using the same direct method framework. Similar to Example 1, Figure 4 shows an excellent agreement between the numerical and exact solutions over the computational domain. The corresponding absolute error distribution, shown in Figure 5, remains uniformly small, while Figure 6 illustrate the relationship between the numerical error and the adaptive step size. The observed behavior confirms the stability, convergence, and accuracy of the proposed method for different problem settings.

Figure 1. Phang DMVS vs Exact Solution

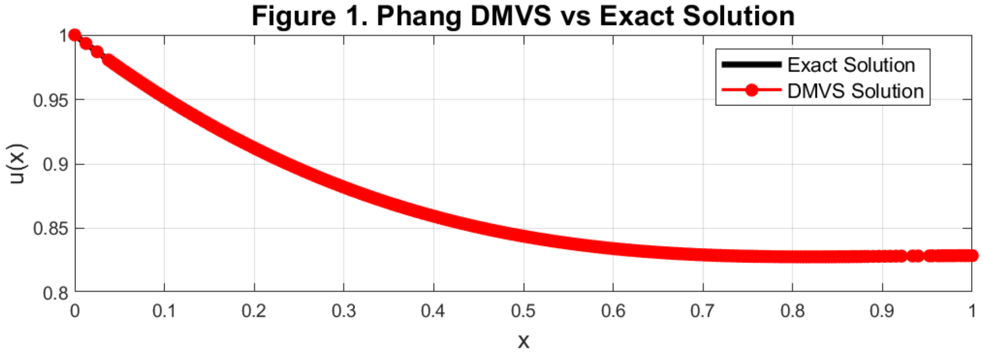


Figure 2. Absolute Error ($L\infty$ = 8.57e-05)

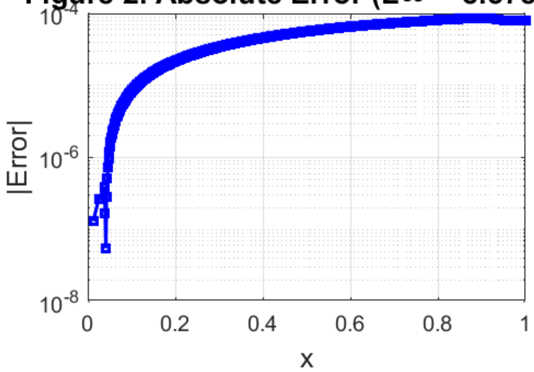


Figure 3. Error vs Step Size

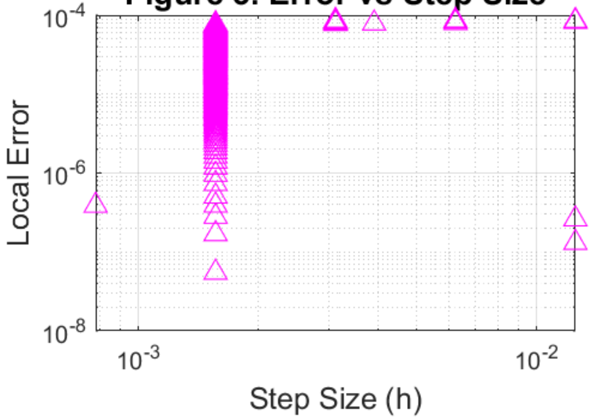


Figure 4. Phang DMVS vs Exact Solution

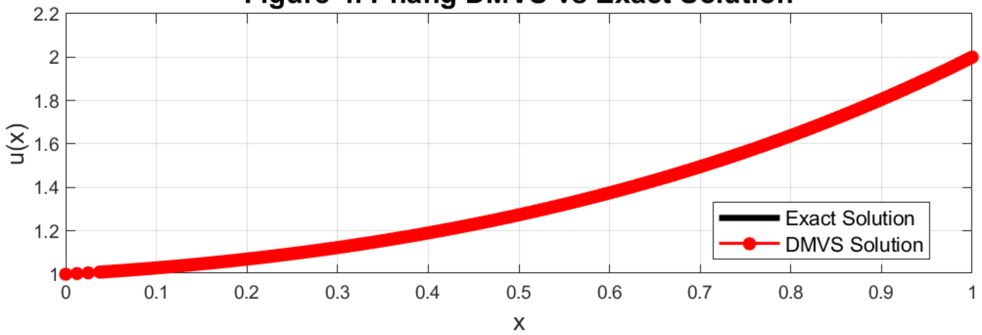


Figure 5. Absolute Error ($L\infty$ = 1.58e-04)

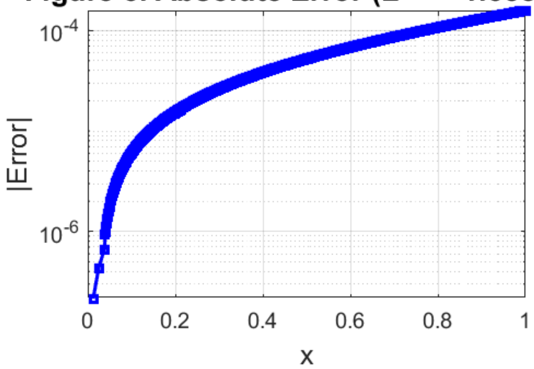
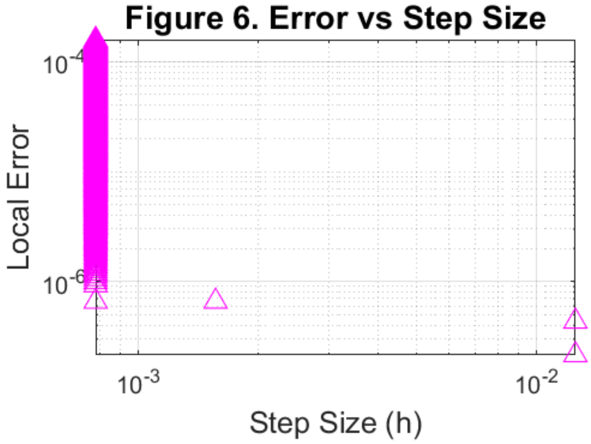


Figure 6. Error vs Step Size



5 Comparison of Both Methods

5.1 Accuracy Comparison

The accuracy of the Galerkin and direct shooting-DMVS methods is evaluated by comparing their numerical solutions with the exact solution.

Let $u(x)$ be the exact solution of BVP. Let $u_G(x)$ and $u_D(x)$ denote the numerical solutions obtained by the Galerkin method and the direct method, respectively.
Define absolute errors:

$$E_G(x_i) = |u(x_i) - u_G(x_i)|$$

and

$$E_D(x_i) = |u(x_i) - u_D(x_i)|.$$

The maximum error norms are given by

$$\|E_G\|_\infty = max_i E_G(x_i) \; and \; \|E_D\|_\infty = max_i E_D(x_i),$$

$$\|E_G\|_2 = \left(\sum_i E_G(x_i)^2\right)^{\frac{1}{2}}, \; \|E_D\|_2 = \left(\sum_i E_D(x_i)^2\right)^{\frac{1}{2}}.$$

For Example 1, the pointwise absolute error obtained by the direct numerical method is of order $\mathcal{O}(10^{-6})$, while the Galerkin approximation exhibits errors of order $10^{-4} - 10^{-5}$, the direct method achieves errors of order 10^{-6}, indicating an order of magnitude improvement in accuracy. It is observed that

$$\|E_D\|_\infty \approx \mathcal{O}(10^{-6}) < \|E_G\|_\infty \approx \mathcal{O}(10^{-4})$$

and

$$\|E_D\|_2 < \|E_G\|_2.$$

This improvement is mainly attributed to the adaptive step-size control employed in the direct method.

Similar accuracy trends are observed for Example 2, where the direct method consistently produces smaller pointwise and global error compared to the Galerkin approach.

While the L_∞ norm highlights the maximum pointwise deviation, the L_2 norm reflects the global distribution of the error. It is observed that the L_2 error associated with the direct shooting-DMVS method is consistently lower than that of the Galerkin approximation, indicating improved accuracy over the entire domain.

5.2 Convergence Behavior

The Galerkin method exhibits convergence through the refinement of basis functions, leading to improved approximation as the number of modes increases.In contrast, the direct method converges through the iterative adjustment of the unknown initial slope using the shooting procedure.

Let h denote the discretization parameter(mesh size or step size). Let $u_G^h(x)$ and $u_D^h(x)$ denote the Galerkin and direct method solution corresponding to step size h.

The numerical methods are said to be convergent if

$$\lim_{h\to 0} \|u(x) - u_G^h(x)\| = 0, \; \lim_{h\to 0} \|u(x) - u_D^h(x)\| = 0.$$

For Galerkin method, $\|u - u_G^h\| \le Ch^p$ and for direct $\|u - u_D^h\| \le Ch^q$. Where $C > 0$ is a constant independent of h, and p and q denote the convergence orders of the Galerkin and direct methods, respectively. From the numerical experiments it is observed that $\|u - u_D^h\|$ decreases more rapidly than $\|u - u_G^h\|$ as $h \to 0$. Direct method converges faster than Galerkin method. This convergence trend is consistently observed in both Example 1 and Example 2.

5.3 Computational Cost

From a computational prespective, the Galerkin method requires the assembly and solution of algebraic syetems, which increases memory usage and computational effort as the number of basis functions grows. The direct shooting-DMVS method aviods matrix assembly and relies only on repeated numerical integration and scalar iterations, making it simpler to implement and computationally efficient for one-dimensional problems.

Let N_G denote the number of basis function used in Galerkin, and N_D denote the number of grid points used in the direct method.

In the Galerkin method, the computational cost mainly arises from

- Assembly of the system matrix
- Evaluation of integrals involving basis functions
- Solution of a linear algebraic system

In the Direct method, the computational cost is associated with

- Numerical integration of the first-order system
- Predictor-Corrector evaluations
- Adaptive step-size control

$$C_D = O(N_D) \; per \; integration \; pass$$

$C_D < C_G$ for comparable accuracy levels. Direct method is computationally more efficient than the Galerkin method.

6 Conclusion

This study presented a direct numerical treatment of a mixed boundary value problem previously analyzed using the Galerkin method. By reformulating the problem through a shooting strategy and employing a DMVS integrator combined with a secant iteration, the unknown initial derivative was successfully determined and the complete solution was obtained. A detailed comparison with the Galerkin-based formulation demonstrates a strong agreement in the solution profiles, initial slope, and error behavior, confirming the consistency of both numerical approaches. The numerical experiments confirmed that both approaches converge to the same physical solution despite their different formulations. It is worth nothing that the Galerkin method provides a stable and reliable approximation framework, while the proposed shooting-DMVS formulation offers an efficient alternative representation of the same physical solution. Overall, the proposed direct shooting-DMVS approach complements the Galerkin framework by offering a simpler and computationally efficient strategy, particularly for one-dimensional mixed boundary value problems.

Acknowledgement

The authors would like to thank Prof. Sanjay Kumar Pandey for his valuable suggestions.

References

[1] J. Crank, *The Mathematics of Diffusion*, Oxford University Press, 1975.

[2] R. L. Burden and J. D. Faires, *Numerical Analysis,* Cengage Learning, 9th Edition, 2011.

[3] S. D. Conte and C. De Boor, *Elementary Numerical Analysis: An Algorithmic Approach,* McGrew-Hill, 1980.

[4] Pavitra, *Laguerre and Hermite polynomials based Galerkin approach for second-order linear ODE for varied BCs using shooting method,* in Role and Application of Applied Mathematics, IIP Series, Iterative International Publishers, 2024.

[5] Phang, C., Ling, L. H., and Ismail, A. I., *Direct method with variable step size via shooting technique for second-order boundary value problems,* World Applied Sciences Journal, vol. 18(2), pp. 129-133, 2012.

[6] Zienkiewicz, O.C., Taylor, R. L., and Zhu, J. Z., *The Finite Element Method: Its Basis and Fundamentals,* 7th ed., Elsevier Butterworth-Heinemann, Oxford, 2013.

[7] T. Kim, D. Kim, K. Hwang, J.J. Sea, *Some identities of Laguerre Polynomials arising from differential equation,* Advances in Difference Equations, 159(2016).

[8] J. N. Reddy, *An introduction to the finite element method,* $3^r d$ Edition, McGrew-Hill, (2011), pp.58-98.

[9] M. K. Jain, *Numerical method for scientific and engineering computation,* Seventh Edition, New Age International, New Delhi (2019).

[10] M. M. Rahman, *Numerical solution of second order boundary value problems by Galerkin method with Hermite polynomials,* Pure and Applied Math Computational, 154(2004), pp.175-182.

[11] M. M. Rahman M. A. Hossen and M.N. Islam, *Numerical solution of second order boundary value problem by Galerkin method with Hermite polynomials,* Annals of Pure and Applied Mathematics, pp.138-148, 2012.

[12] Liu, Y., *Studies on Neumann-type boundary value problems for second-order nonlinear p-Lalacian-like functional differential equations*, Nonlinear Analysis, Real World Applications, 10(2009), pp.333-344.

[13] Ramadan, M. A., I. F. Lashien and W. K. Zahra, *Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems,* Applied Mathematics and Computation, 184(2007), pp. 476-484.

[14] Faires, D. and R. L. Burden, *Numerical Analysis,* USA: International Thomson Publishing Inc, 1998.

[15] Yao, Q., *Successively iterative method of nonlinear Neumann boundary value problems,* Applied Mathematics and Computation, 217(2010), pp. 2301-2306.

[16] Shampine, L. F. and M. K. Gordan, *Computer solution of ordinary differential equations: The initial value problem,* San Franciso, W. H. Freeman and Company, 1975.

Cite this Article:

Pavitra; Pandey, P. (2026). On the Accuracy and Efficiency of Galerkin and Direct Methods for Second-Order Boundary Value Problems with Neumann Boundary Conditions. *Pi International Journal of Mathematical Sciences*, 2(2), 01–7.
Journal URL: https://pijms.com/
DOI: https://doi.org/10.59828/pijms.v2i2.29