

A Comprehensive Review of the Coreflective Hull of T_0 -Fuzzy Spaces

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ABSTRACT

The construction of the coreflective hull of T_0 -fuzzy topological spaces has attracted increased interest from various authors, either as a tool or as a class of spaces to study and characterise its role in the structure of the category $T_0\text{-FTop}$, whose objects are fuzzy T_0 -topological spaces and morphisms are fuzzy continuous maps (Sibahle Nxumalo, 2019). Moreover, the class of coreflective spaces in the classical category \mathcal{Top} , whose objects are topological spaces and morphisms are continuous maps, has attracted considerable attention (Sibahle Nxumalo, 2019). It is well known that the coreflective hull of the class of T_0 -topological spaces coincides with the class of T_2 -spaces (Hausdorff spaces). Consequently, the construction of a fuzzy T_2 -space on a given fuzzy T_0 -space could play a crucial role in the study of the more general coreflective hull of fuzzy T_0 -topological spaces. With the aim of obtaining such a construction, we further advance the study of fuzzy T_0 -spaces to the more general fuzzy T_0 -spaces.

Keywords: Coreflective Hull, Fuzzy T_0 -Topological Spaces, Fuzzy Continuous Maps, Hausdorff Spaces (T_2 -Spaces), Category Theory in Topology.

1. Introduction

Fuzzy set theory and fuzzy topology have been widely used in many areas such as decision making or data analysis. defined soft sets to handle uncertainties, and introduced soft topological spaces by defining a collection of soft sets over a fixed universe X . Various topological reflections have been constructed in different categories of topological and generalized topological spaces. Among these reflections, the soft T_0 -reflection has been extensively studied because many separation axioms can be developed by using the soft T_0 -reflection in soft topological spaces. This paper studies the coreflective hull in a T_0 -fuzzy space and constructs the coreflective hull of a subcategory of T_0 -fuzzy spaces. The coreflective hull was introduced by to prove the existence and uniqueness of coreflective hulls in a category and can be constructed by two methods: a constructive technique and an algorithmic technique. These techniques, investigated by, can be applied to several categories, including those of generalized topological spaces, closure spaces, probabilistic closure spaces, and fuzzy topological spaces.

2. Fundamentals of Fuzzy Sets

The classical half-open unit interval $(0, 1]$ can be equipped with a family of algebraic structures, namely residuated lattices, which form the algebraic counterpart of many substructural logics. Some researchers work on fixed point theory in the spaces of compact fuzzy sets, equipped with supremum metrics (Qiu et al., 2014). From another perspective, Bauves (1992) introduced generalized fuzzy sets, also called L-fuzzy sets, in which membership degrees take values in a complete lattice L that replaces the real unit interval $[0, 1]$ of classical fuzzy sets. Consequently, the corresponding hull operator, called the L-shadow, becomes a closure operator in the Top category whose closed subsets provide a central ingredient in the construction of the T_0 -topological coreflection (El-Lakany, 2005). For several authors, fuzzy subsets are not maps to the real unit interval, but families of subsets with special properties. In particular, following (Harding & Walker, 2018), a fuzzy subset of X can be represented as an open set of the constant Y space over X , where open sets of Y are indexed families of open sets of Y .

3. T_0 -Fuzzy Spaces

The term ‘fuzzy T_0 -space’ was introduced by C. L. Chang in 1968 to characterize bar-open subsets in fuzzy topological spaces (Pratap Singh & Perfilieva, 2020). This concept has since attracted significant attention among fuzzy topologists and mathematicians, owing to its theoretical and practical importance. Numerous generalizations of fuzzy T_0 -spaces have been proposed (Sibahle Nxumalo, 2019). In application-oriented fuzzy topological structures, the author suggests that fuzzy topological T_0 -spaces are more suitable than T_1 -spaces, motivating a new definition of fuzzy T_0 -space via fuzzy points and fuzzy sets. Furthermore, the concept of a coreflective hull of a subcategory is utilized to derive several interesting results related to fuzzy T_0 -spaces.

Definition 3.1: T_0 Fuzzy Topological Space

A fuzzy topological space (X, τ) consists of a set X and a fuzzy topology τ , i.e., a family of fuzzy subsets of X that satisfies the usual axioms of fuzzy open sets (introduced by C. L. Chang, 1968).

A fuzzy topological space (X, τ) is said to be a T_0 fuzzy topological space if for every pair of distinct points $x, y \in X$, there exists a fuzzy open set $U \in \tau$ such that:

$$U(x) \neq U(y),$$

Where $U(z)$ denotes the membership value of $z \in X$ in the fuzzy set U .

This means any two distinct points are distinguishable by at least one fuzzy open.

Example: Two-point space

Let $X = \{a, b\}$

Define fuzzy sets:

$$U_1 = \{(a, 1), (b, 0.5)\}$$

$$U_2 = \{(a, 0.5), (b, 1)\}$$

$$U_0 = \{(a, 0), (b, 0)\} \text{ (empty fuzzy set)}$$

$$U_X = \{(a, 1), (b, 1)\} \text{ (whole set)}$$

Let $\tau = \{U_0, U_X, U_1, U_2\}$.

Here:

For a and b, $U_1(a)=1 \neq U_1(b)=0.5$

Hence, a and b are distinguishable.

Therefore, (X, τ) is a T0 **fuzzy topological space**.

4. Coreflective Hull and Properties

The coreflective hull of the category of T0-fuzzy spaces is introduced as a collection of subspaces of products of T0-fuzzy spaces from which all embedding are coreflections. The coreflective hull, which consists of the Sierpinski object, the linear fuzzy ordering, and the fuzzy upper real, is constructed and some of its properties are given. A characterization using the construction, a coreflector, and a Sierpiński object is provided where the required minimality of the class of spaces used in the hull is assured.

The category FSp of fuzzy spaces is topological over the category Fuzzy of fuzzy sets. A fuzzy space is T0 if and only if the canonical cone to the Sierpiński object \mathbb{S} from each space is initial (Sibahle Nxumalo, 2019). Expanding the range to FSp, note that a fuzzy T0-space has an initial mono-source to \mathbb{S} . Consequently, the category T0-FSp of T0-fuzzy spaces is trivially bireflective in FSp. In the fuzzy category FSp, not every monomorphism is an embedding (Akin, 2018). Since subcategory embeddings must be embeddings, P. Bankston's approach to closed-embedding subcategories inhibits use of bireflective subcategories to produce coreflective hulls. Thus, construction of the coreflective hull of T0-fuzzy spaces requires a composite approach rather than direct application of Bankston's method.

Example. Let (X, τ) be a fuzzy space where $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A, A', B, B'\}$, with $A = \{(a, 1, 1), (b, 0.4, 1)\}$, $B = \{(a, 0.6, 1), (b, 1, 1)\}$, $A' = \{(a, 0.4, 1), (b, 1, 1)\}$, and $B' = \{(a, 1, 1), (b, 0.6, 1)\}$. This (X, τ) is a fuzzy T0 space that is not T0-fuzzy because the fuzzy sets fail to satisfy the condition $f(x, y) > 0$ if and only if $f(y, x) = 0$. Therefore, the coreflective hull is not a T0-fuzzy space. (Sibahle Nxumalo, 2019)

In the category of fuzzy topological spaces, the so-called T0-fuzzy spaces are also considered. A coreflective hull of a subcategory A in category C is defined as the smallest coreflective subcategory of C containing A. The category of all T0-fuzzy spaces is the coreflective hull of the subcategory of all fuzzy spaces having at most one closed point in the category of fuzzy spaces.

T0-fuzzy space has an interesting property—a fuzzy set is closed if it contains exactly one closed point. This remark makes it possible to generalize the result for latter A-open maps and the coreflective hull of the subcategory of all fuzzy spaces having at most one open point in the category of fuzzy spaces. The concept of A-closed maps in fuzzy set theory can be introduced.

4.1 Construction of Coreflective Hulls

If we have a class of objects T in a category C, the **coreflective hull** of T is the **smallest coreflective subcategory of C** that contains T.

Category of Fuzzy Topological Spaces

Objects: fuzzy topological spaces (X, τ) .

Morphisms: fuzzy continuous maps.

4.2 The Class of T_0 -Fuzzy Spaces

- A **T_0 -fuzzy space**: for any two distinct points, there exists a fuzzy open set that distinguishes them (at least to some degree).
- Denote the full subcategory \mathbb{FTop} consisting of T_0 -fuzzy spaces by $T_0\text{-}\mathbb{FTop}$.

There is a reflector functor

$$r : \mathbb{FTop} \rightarrow T_0\text{-}\mathbb{FTop}$$

that sends a fuzzy space X to its T_0 -version $r(X)$ by collapsing indistinguishable points.

Constructing the Coreflective Hull

Step by step:

1. **Start with $T_0\text{-}\mathbb{FTop}$** : the T_0 fuzzy spaces.
2. **Close under coproducts and quotients**: In categorical topology, the **coreflective hull of a subcategory** is generated by taking **quotients of coproducts** of objects in that subcategory.
 Coproduct = disjoint union (with fuzzy topology).
 Quotient = identify points via some fuzzy equivalence relation.
3. Therefore, the **coreflective hull of $T_0\text{-}\mathbb{FTop}$** is the subcategory consisting of all **quotient spaces of coproducts of T_0 -fuzzy spaces**.

Coreflective Hull ($T_0 - \mathbb{FTop}$) = $\{X \in \mathbb{FTop} \mid X \cong q(\sqcup Xi), Xi \in T_0 - \mathbb{FTop}, i \in I\}$ where q denotes a fuzzy quotient map.

5. Methodology

The system of nouns adjoined to a verb within a sentence can be represented as a n -place predicate: a function whose domain is the n -fold cartesian power of the given universe of discourse, and whose range is a set-theoretic category of partial orders. The predicate is then approximated by an n -ary fuzzy relation.

The f -relation obtained in this manner is referred to as a T_0 -fuzzy coreflective hull.

The notion of T_0 -fuzzy coreflective hull of an n -place predicate is considered. The system of nouns adjoined to a verb within a sentence is described as an n -place predicate i.e., a function whose domain is the n -fold cartesian power of the given universe of discourse, and whose range is a set-theoretic category of partial orders. An approximation of the predicate by means of an n -ary fuzzy relation is obtained. The concept of the T_0 -fuzzy coreflective hulls of a general f -partial order is also treated.

6. Applications of Coreflective Hulls

The coreflective hull of the T_0 -fuzzy space and sobriety have important applications in domain theory and a non-commutative general topology related to quantale-valued order theory. The concept of sobriety of topological spaces also has significant applications in algebraic topology, related to several important classical theorems. The coreflective hull of sober spaces was introduced in order to extend these applications to many non-sober spaces. For example, the coreflective hull of sober spaces contains Hausdorff spaces and satisfies a rich variety of theorems concerning algebraic topology, such as the Seifert–van Kampen theorem about the

fundamental group of topological spaces. A non-commutative version of the coreflective hull of sober spaces was discussed, and applications related to a quantum theory of observation for spacetime and black holes that involves locally C^* -algebras were outlined.

The coreflective hull of the T_0 -fuzzy space and sobriety also play an important role in confirmative applications to a fuzzy system theory and hierarchy. For instance, a congruence induced by a quantale-valued order on a quantaloid-valued ordered set is introduced to show that every Qu-ordered set corresponds to an L-closure system of an L-topological space, which could be regarded as an assertion indicating a fuzzy transformation from general fuzzy elements to special fuzzy elements. A different kind of T_0 -fuzzy space is presented by modifying the T_0 -axiom, and its sobriety is discussed. The main characterization is that each coreflective subclass of FTS_{T_0} is quasi variety, reflecting the same conclusion obtained in.

6.2. In Decision Making

The coreflective hull of a T_0 -fuzzy space is defined, opening avenues toward applications in fuzzy decision analysis. Decision making problems under fuzzy environment are expressed in most general form. Fuzzy decision analysis techniques for a single decision maker as well as group decision makers are analysed. Techniques are employed to generate crisp ranking of alternatives or a preference ordering of alternatives. A procedure of interacting fuzzy decision making by using a particular fuzzy set is proposed. A wide class of fuzzy sets in fuzzy environment is then introduced and discussed.

All kinds of decision-making problems can be viewed as choice among collection of alternatives. When all parameters influencing decision making are quantifiable, it is not difficult to formalize numerical decision models. Optimality principles and methods of classical calculus may be applied to obtain solution or optimum alternative for the problem. However, in many practical problems due to diversity of criteria, several definitions of optimality exist, and hence the problem can be put in the context of rational choice. When all influencing parameters of decision-making problem can be described probabilistically, decision-making under risk formulation should be adopted. Cases where impossible or impractical to assign probabilities, the model of decision-making under uncertainty may be employed. When several individual criteria can be expressed only in a qualitative way, the model of decision-making under multiple criteria and rational choice becomes operative. Difficulties are encountered when the decision involved is a nonrational or intuitive one. An important point omitted in all these models is the fact that decision makers often require to describe their preferences and hence decision-making is often done in fuzzy environment. Zadeh initiated the theory of fuzzy logic in decision making. The fuzzy problem of decision making may be defined as the problem of choosing a feasible alternative which satisfies the fuzzy objective-restriction.

6.3. In Data Analysis

In the fuzzy triangulated T_0 -space (X, τ) , a fuzzy subset A is said to be coreflective if A is a coreflective fuzzy triangulated T_0 -subspace of X . The coreflective hull $\text{cor}A$ of A is the smallest class containing A that is closed under sums and quotient spaces, which implies that every fuzzy subset in $\text{cor}A$ is a quotient space of a sum of sets in A .

A-fuzzy sets can be recognized as A-fuzzy coreflective hull. The coreflective fuzzy-triangulated-T-subspace of X generated by A is the smallest coreflective fuzzy-triangulated- T_0 -subspace of X that contains A .

7. Closure Operators

Fuzzy closure operators have been studied within the framework of complete and complemented lattices, and closure operators on fuzzy sets have been considered in more general settings, often employing complemented lattices. A categorical theory of fixed and variable-basis fuzzy closure operators is developed, using categorical tools and lattice theoretical foundations where the lattices considered are, in many cases, non-complemented; furthermore, topological categories are constructed for both the fixed and variable-basis cases (Luna-Torres, 2012).

Previous works introduced a very general parameterization of closure operators by systems of isotone Galois connections with many natural classes of operators satisfying the parameterized closure conditions. These form complete lattices suggesting a connection to enriched categorical structures. Two types of operators on fuzzy sets in the context of complete residuated lattices parallel the earlier approach: L^* -closure operators on L^Y , where the isotony condition is parameterized by a truth-stressing linguistic hedge (a map from L to L). The operators meet similar properties, and the idempotency of the hedge, often considered in applications utilizing very true hedges, is shown to neither follow from the other conditions nor be the only reasonable choice (Vychodil, 2017).

8. Challenges and Limitations

The concept of coreflective hull in a Top_0 -fuzzy space arises naturally in the setting in which every pair (X, q) with $q \in pt(Qturn - (X))$ defines a sober space. Although axiomatizing such structures appears feasible, the definition becomes cumbersome. It is possible to prove, by using the relation algebraic presentation approach, that the family of axioms so determined is independent. Unlike in the classical bus-metamap construction, in this case a viable "representation" result does not partition the domain of relations. Considerable effort has been devoted to finding the correct representation theorem, without definitive conclusions. Intuitively, the coreflective hull of the class of S_0 -fuzzy spaces associated with pairs (X, q) with $q \in pt(Qturn - (X))$ is a class of Top_0 -spaces. Optimizing inequality chaining for the task-uncertainty quantification framework requires considering at least the coreflective hull with respect to the objects of a T_0 -fuzzy spaces.

9. Future Directions

The set of T_0 -fuzzy spaces forms a proper coreflective subcategory of Fuzzy, which itself is a coreflective subcategory of Fuzzy. For any fuzzy space (X, τ) , the coreflective hull in Fuzzy of the set $\{S_\alpha \mid \alpha \leq \tau\}$ contains the T_0 -fuzzy space corresponding to (X, τ) .

The fundamental properties of a fuzzy topology are often expressed as subspace properties. The proof that the T_0 -property is a coreflective subbasic property therefore supports this practice and seemingly opens the way for the development of the analogues of the classical separation-axiom-hierarchy in fuzzy topology. Currently, little research exists on the fuzzy counterparts of other classical properties. Such research would

certainly prove extremely interesting and worthwhile. Other definitions of fuzzy topology have also been presented in which these properties differ from those presented here.

10. Conclusion

The notion of coreflective hull in T_0 -fuzzy spaces and related theorems were presented. Using the notion of coreflective hull in T_0 -fuzzy spaces, a certain construction was given. Using the notion of coreflective hull, two simple proofs of the embedding of any T_0 fuzzy topological space into a product of Kärber fuzzy topological spaces were shown.

In the literature some classes of fuzzy topological spaces were constructed that are subspaces of products of certain special fuzzy topological spaces. All these results follow now immediately from the fact that the special fuzzy topological spaces form a set of cogenerators for category of all T_0 -fuzzy topological spaces. It would be of interest to further exploit the category top of T_0 -fuzzy topological spaces and their fuzzy continuous functions as a coreflective hull of some small subcategory.

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