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On Various Forms of Solution of k-Hypergeometric Differential Equations

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Abstract

In this paper, the series solutions of the second order k-hypergeometric differential equation based on the Frobenious method have been discussed. First, we obtained two linearly independent solutions and then the two derived forms each valid in their respective complex region characterize by the regular singular points of this differential equation. The main objective of the paper is to present the existence of various forms of solutions of the k-hypergeometric differential equation valid in different regions of the complex z-plane.

Keywords: *k-hypergeometric equations; Forbenious method; k-hypergeometric series solutions, regular singular point.*

Mathematics Subject Classification (MSC) 2020: *33C05, 34A25.*

1. Introduction

The Gauss hypergeometric equation which is expressed by

$$z(1-z)\frac{d^2y}{dz^2}+[c-(1+a+b)z]\frac{dy}{dz}-aby=0 \tag{1}$$

with 3 regular singularities {0, 1, ∞} has been extensively studied by various authors including Coddington[3], Campos [4], Gasper [6], RainVille [11], Slater [12], Whittaker [14].

A hypergeometric series solution ${}_2\mathcal{F}_1[a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$

of (1) can be derived by the Frobenius method is convergent in the region.

$R=\{z\colon |z|<1\}, when\ c\notin z, z^{1-c}2F1[a-c+1, b-c+1; 2-c; z]$ is also a linearly independent solution of (1) and convergent in the region R.

Kummer discuss the existence of twenty-four solutions Slater [12], which provide complete solutions of (1) valid in any region of complex z-plane. Let us consider a k-hypergeometric equation of the form:

$$kz(1-kz)\frac{d^2y}{dz^2}+[c-(k+a+b)kz]\frac{dy}{dz}-aby=0 \tag{2}$$

This differential equation is called the homogenous k-hypergeometric equation, which has been defined only in recent years and has limited studies. The aim of this paper is to find out the different form of k-hypergeometric series solutions, valid in different region of the complex z- plane.

2. Preliminaries

In this section, we briefly review some basic definitions and facts concerning the Fuchsion type differential equation and k-hypergeometric series. Some surveys and literature for k-hypergeometric series and the k- hypergeometric differential equation can be found in Diaz et al. [5], Krasniqi [7], Mubeen [8-10], Shengfeng [13] and Yilmazer [15].

***Definition 1*:** Assume that p(z) and q(z) be two complex valued functions. Let a second order differential equation in standard form:

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y(z) = 0 \qquad (3)$$

Then the method about finding an infinite series solution of equation (3) is called the Frobenius method.

***Definition 2*:** If $z = z_0$ is a singularity of (3), then $z = z_0$ is a regular singularity of (3) if and only if $(z - z_0)p(z), (z - z_0)^2 q(z)$ are analytic in $\{z: |z - z_0| < R\}$, where R is a positive real number.

***Definition 3*:**

The Pochhammer k-symbol $(\sigma)_{n,k}$ is defined by:

$$(\sigma)_{n,k} = \sigma(\sigma + k)(\sigma + 2k) \dots (\sigma + (n-1)k), \sigma \neq 0 \qquad (4)$$

$(\sigma)_{0,k} = 1$ where $k > 0$

***Definition4*:** The k-hypergeometric series with three parameters a, b, and c is defined as:

$${}_2\mathcal{F}_{1,\mathrm{k}}[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} \qquad (5)$$

3. Theorem

The homogeneous k-hypergeometric differential equation (2) has a general solution in the form

$$y(z) = \mathrm{A}\,{}_2\mathcal{F}_{1,\mathrm{k}}[(a,k),(b,k);(c,k);z] + B\, z^{1-\frac{c}{k}}\,{}_2\mathcal{F}_{1,\mathrm{k}}[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z]$$

for $|z| < \frac{1}{k}, k \in R^+, a, b, c \in R$ and 2k-c neither zero nor negative integer.

Proof: Let us assume that

$$y = \sum_{r=0}^{\infty} u_r z^{l+r} \ (where\ u_0 \neq 0)$$

Is any solution of the equation (2). Then by direct differentiation of the series, we have

$$kz(1-kz)\sum_{r=0}^{\infty}(l+r)(l+r-1)u_r z^{l+r-2} + [c-(a+b+k)kz]\sum_{r=0}^{\infty}(l+r)u_r z^{l+r-1} - ab\sum_{r=0}^{\infty}u_r z^{l+r} = 0 \qquad (6)$$

$$\sum_{r=0}^{\infty}k(l+r)(l+r-1)u_r z^{l+r-1} + \sum_{r=0}^{\infty}c(l+r)u_r z^{l+r-1} - \sum_{r=0}^{\infty}k^2(l+r)(l+r-1)u_r z^{l+r} - \sum_{r=0}^{\infty}k(a+b+k)(l+r)u_r z^{l+r} - ab\sum_{r=0}^{\infty}u_r z^{l+r} = 0 \qquad (7)$$

$$kz^l\sum_{r=1}^{\infty}(l+r)(l+r-1)u_r z^{r-1} + cz^l\sum_{r=1}^{\infty}(l+r)u_r z^{r-1} - k^2 z^l\sum_{r=1}^{\infty}(l+r-1)(l+r-2)u_{r-1}z^{r-1} - k(a+b+k)z^l\sum_{r=1}^{\infty}(l+r-1)u_{r-1}z^{r-1} - ab\, z^l\sum_{r=1}^{\infty}u_{r-1}z^{r-1} + u_0 l(k(l-1)+c)z^{l-1} = 0 \qquad (8)$$

Hence by comparing the coefficient on both sides of above equation, we must have as the indicial equation

$$l[k(l-1)+c]=0 \tag{9}$$

and the difference equation

$$(l+r+1)[k(l+r)+c]u_{r+1}=[k(l+r)+a][k(l+r)+b]u_r \tag{10}$$

for $r=0,1,2,\ldots$.

On solving the indicial equation (9), we have $l=0$ and $l=1-\frac{c}{k}$

Case I: $l=0$, From equation (10), we have the solution of equation (2):

$$y_1(z)=u_0\,{}_2\mathcal{F}_{1,k}[(a,k),(b,k),(c,k);z] \tag{11}$$

Provided that c is neither zero nor a negative integer.

Case II: $l=1-\frac{c}{k}$. From equation (10), we have a difference equation

$$(r+1)(2k-c+kr)u_{r+1}=(a+k-c+kr)(b+k-c+kr)u_r \tag{12}$$

and equation (12) gives a second solution

$$y_2(z)=\ u_0z^{1-\frac{c}{k}}\ {}_2\mathcal{F}_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] \tag{13}$$

Provided that c is not a positive integer $\geq 2k$

Hence one complete solution of equation (2) is

$$y(z)=\ A\ {}_2\mathcal{F}_{1,k}[(a,k),(b,k);(c,k);z]+B\ z^{1-c/k}\ {}_2\mathcal{F}_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] \tag{14}$$

Where $|z|<\frac{1}{k}$ and A, B are constant.

4. Forms of the solutions of the equations (2) valid in the region |z| <1/k.

$$\text{Now let } y(z)=\left(\frac{1}{k}-z\right)^s w(z) \tag{15}$$

$$y'=\frac{1}{k^s}\left[(1-kz)^s\frac{dw}{dz}-sk(1-kz)^{s-1}w\right]$$

$$y''=\frac{1}{k^s}\left[(1-kz)^s\frac{d^2w}{dz^2}-2sk(1-kz)^{s-1}\frac{dw}{dz}+sk^2(s-1)(1-kz)^{s-2}w\right]$$

On plunging everything into the equation (2) yields the equation (16)

$$kz(1-kz)w''+[c-(a+b+k+2sk)kz]w'+\left[\frac{s(s-1)k^3z-sk\{c-(a+b+k)kz\}}{(1-kz)}-ab\right]w=0 \tag{16}$$

This equation is also of k-hypergeometric type if $(1-kz)$ exactly divides $s(s-1)k^3z-sk\{c-(a+b+k)kz\}$, that is, if either $s=0$ or $s=\frac{1}{k}(c-a-b)$.

When $s=0$ the two solutions (11) and (13) are given; but when $s=\frac{1}{k}(c-a-b)$, then two new solutions are given, valid in the region $|z|<\frac{1}{k}$.

These are

$$y_3(z)=(\tfrac{1}{k}-z)^{\frac{1}{k}(c-a-b)}\,{}_2\mathcal{F}_{1,k}[(c-a,k),(c-b,k);(c,k);z] \tag{17}$$

and

$$y_4(z)=z^{1-\frac{c}{k}}(\tfrac{1}{k}-z)^{\frac{1}{k}(c-a-b)}\,{}_2\mathcal{F}_{1,k}[(k-a,k),(k-b,k);(2k-c,k);z] \tag{18}$$

5. Forms of the solutions of the equations (2) valid in the region |z-1| <1/k.

let $t=\frac{1}{k}-z$, then

$$y'=\frac{dy}{dz}=-\frac{dy}{dt}=-\dot{y}\ and\ y''=\ddot{y}$$

Thus $kz(1-kz)y''+[(c-(k+a+b)kz]y'-aby=0$ becomes

$$kt(1-kt)\ddot{y}+[(k+a+b-c)-(k+a+b)kt]\dot{y}-aby=0$$

This is in from of k-hypergeometric equation satisfied by

$$y=\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(k+a+b-c,k);t]$$

Hence four further solution of the k-hypergeometric equation (2) are

$$y_5 = {}_2\mathcal{F}_{1,k}\left[(a,k),(b,k);(k+a+b-c,k);\frac{1}{k}-z\right] \tag{19}$$

$$y_6 = \left(\frac{1}{k}-z\right)^{\frac{1}{k}(c-a-b)}{}_2\mathcal{F}_{1,k}\left[(c-a,k),(c-b,k);(k-a-b+c,k);\frac{1}{k}-z\right] \tag{20}$$

$$y_7 = z^{1-\frac{c}{k}}{}_2\mathcal{F}_{1,k}\left[(k+a-c,k),(k+b-c,k);(k+a+b-c,k);\frac{1}{k}-z\right] \tag{21}$$

and

$$y_8 = z^{1-\frac{c}{k}}\left(\frac{1}{k}-z\right)^{\frac{1}{k}(c-a-b)}{}_2\mathcal{F}_{1,k}\left[(k-a,k),(k-b,k);(k-a-b+c,k);\frac{1}{k}-z\right] \tag{22}$$

These solution hold in the region $|z-1| < \frac{1}{k}$, provided a, b and c are neither negative integers nor zero.

Conclusion:

In this paper, we investigate the four solutions of k-hypergeometric differential equations in each of the region $|z| < 1/k$ and in the region |z-1| < 1/k . By combining the solutions of the said regions, we may derived other forms of solutions, for the Gauss hypergeometric differential equation valid in some other region of the complex z- plane. Since the results investigated here are general in nature, it is expected that the results will be useful addition in the theory of the Gauss hypergeometric functions. It would be interesting to have more research about this case.

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